Sample Midterm Answers

1. (a) As x gets close to a, $f(x)$ gets close to L.

   (b) $f(x)$ is cts at a if
$$\lim_{x\to a} f(x) = f(a).$$

   (c) The derivative of $f(x)$ at a is
$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

2. (a) The function is not cts at $x = 1$ and $x = 6$.

   (b) The derivative does not exist at $x = 1, x = 6, x = -2, x = -4$.

3. The slope of the secant line is
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(3) - f(2)}{3 - 2} = \frac{[3(3)^2 - 3] - [3(2)^2 - 2]}{1} = 16$$

4. (a)
$$\lim_{x\to 2} x^3 - x^2 + 3x - 2 = (2)^3 - (2)^2 + 3(2) - 2 = 12$$

   (b)
$$\lim_{x\to -2} \frac{x^2 - 4}{x^2 + 5x + 6} = \lim_{x\to -2} \frac{(x-2)(x+2)}{(x+3)(x+2)} = \lim_{x\to -2} \frac{x-2}{x+3} = \frac{(-2) - 2}{-2 + 3} = -4$$

   (c)
$$\lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3} = \lim_{x\to 3} \frac{\frac{3-x}{3x}}{x - 3} = \lim_{x\to 3} \frac{-(x-3)}{(3x)(x-3)} = \lim_{x\to 3} \frac{-1}{3x} = \frac{-1}{9}$$

5. (a)
$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^3 - 1}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{h^3+3h^2+3h+1-1}{h} \\
&= \lim_{h\to 0} \frac{h^3+3h^2+3h}{h} \\
&= \lim_{h\to 0} h^2+3h+3 \\
&= 3
\end{aligned}
$$

(b)

$$
\begin{aligned}
f'(2) &= \lim_{x\to 3} \frac{f(x)-f(2)}{x-3} \\
&= \lim_{x\to 3} \frac{\sqrt{x-1}-\sqrt{2}}{x-3} \\
&= \lim_{x\to 3} \frac{\sqrt{x-1}-\sqrt{2}}{x-3}\left(\frac{\sqrt{x-1}+\sqrt{2}}{\sqrt{x-1}+\sqrt{2}}\right) \\
&= \lim_{x\to 3} \frac{(x-1)-(2)}{(x-3)(\sqrt{x-1}+\sqrt{2})} \\
&= \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x-1}+\sqrt{2})} \\
&= \lim_{x\to 3} \frac{1}{\sqrt{x-1}+\sqrt{2}} \\
&= \frac{1}{\sqrt{2}+\sqrt{2}} \\
&= \frac{1}{2\sqrt{2}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f'(4) &= \lim_{x\to 4} \frac{f(x)-f(4)}{x-4} \\
&= \lim_{x\to 4} \frac{\frac{1}{x+2}-\frac{1}{6}}{x-4} \\
&= \lim_{x\to 4} \frac{\frac{6-(x+2)}{6(x+2)}}{x-4} \\
&= \lim_{x\to 4} \frac{-x+4}{6(x+2)(x-4)}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{x\to 4} \frac{-(x-4)}{6(x+2)(x-4)} \\
&= \lim_{x\to 4} \frac{-1}{6(x+2)} \\
&= \frac{-1}{36}
\end{aligned}
$$

6. (a) $f'(x) = \frac{1}{2}(5x-3)^{-\frac{1}{2}}(5) = \frac{5}{2\sqrt{5x-3}}$

   (b) $f'(x) = 2(\cos x)(-\sin x) = -2\cos x \sin x$

   (c) $f'(x) = \cos(x^{\frac{2}{3}})(\frac{2}{3}x^{\frac{-2}{3}})$

   (d) $f'(x) = \tan x(2e^{2x}) + \sec^2 x(e^{2x}) = e^{2x}(2\tan x + \sec^2 x)$

   (e)

$$
\begin{aligned}
f'(x) &= \frac{(x^3+3)(2x)-(3x^2)(x^2+2)}{(x^3+3)^2} \\
&= \frac{(2x^4+6x)-(3x^4+6x^2)}{(x^3+3)^2} \\
&= \frac{-x^4-6x^2+6x}{(x^3+3)^2} \\
&= \frac{-x(x^3-6x+6)}{(x^3+3)^2}
\end{aligned}
$$

   (f)

$$f'(x) = \tan'(6x)(6x)' = \frac{1}{\sqrt{1-(6x)^2}}(6) = \frac{6}{\sqrt{1-36x^2}}$$

   (g)

$$f'(x) = \ln'(x^{-4}+10x^3)(x^{-4}+10x^3)' = \frac{-4x^{-5}+30x^2}{x^{-4}+10x^3)}$$

   (h)

$$f'(x) = \sec'(\ln x)(\ln x)' = \sec(\ln x)\tan(\ln x)\frac{1}{x}$$

7. First, we find the derivative:

$$f'(x) = (x^2+3)\left(\frac{1}{2\sqrt{x}}\right) + (2x)(\sqrt{x}+1)$$

Then we substitue $x = 4$ to find the slope of the tangent line:

$$f'(4) = (4^2+3)\left(\frac{1}{2\sqrt{4}}\right) + 2(4)(\sqrt{4}+1) = (19)\left(\frac{1}{8}\right) + 8(3) = \frac{275}{8}$$

We then substitute $x = 4$, $m = \frac{275}{8}$, and $y = f(4) = (4^2+3)(\sqrt{4}+1) = 171$ to solve for b:

$$y = mx + b \Rightarrow (171) = \frac{275}{8}(4) + b \Rightarrow b = \frac{67}{2}$$

Thus the equation of the tangent line at $x = 4$ is $y = \frac{275}{8}x + \frac{67}{2}$.

8. First, we find the derivative using implicit differentiation:

$$(2y)(y') = 20x^3 - 2x$$

Then solve for y':

$$y' = \frac{10x^3 - x}{y}$$

Then substitute $x = 1$, $y = 2$:

$$y' = \frac{10(1)^3 - 1}{2} = \frac{9}{2}$$

Thus the slope of the tangent line at $(1, 2)$ is $\frac{9}{2}$.

9. The car is at rest when $f'(t) = 0$. So we have:

$$\begin{aligned} \frac{1}{t^3 - 6t^2 + 12t + 1}(3t^2 + 12t + 12) &= 0 \\ 3t^2 - 12t + 12 &= 0 \\ t^2 - 4t + 4 &= 0 \\ (t-2)(t-2) &= 0 \end{aligned}$$

So the only solution is when $t = 2$. Thus the car is at rest when $t = 2$.

10. We can re-write the equation as $F = GmMr^{-2}$. Now, each of G, m, and M are constants, so when we take the derivative, we can ignore them. So we have:

$$\frac{dF}{dr} = -2GmMr^{-3} = \frac{-2GmM}{r^3}$$

We then substitute $m = 10$, $M = 3$, and $r = 2$ to get

$$\frac{dF}{dr} = \frac{-2G(10)(3)}{2^3} = \frac{-60G}{8} = \frac{-15G}{2}$$

Thus the rate of change is $\frac{-15G}{2}$.

11. We know that $\frac{dV}{dt} = \frac{1}{4}$, and we want to find $\frac{dh}{dt}$. First of all, since the height is always twice the radius, we know $r = \frac{h}{2}$, so we can re-write the volume equation:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

Then we take the derivative with respect to time t:

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\left(\frac{dh}{dt}\right)$$

We then solve for our required quantity, $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\left(\frac{dV}{dt}\right)$$

Finally, we substitute $h = 2$ and $\frac{dV}{dt} = \frac{1}{4}$ to get

$$\frac{dh}{dt} = \frac{4}{\pi(2)^2}\left(\frac{1}{4}\right) = \frac{1}{4\pi}$$

Thus the rate of change of height is $\frac{1}{4\pi}$.

12. Let x represent the distance the south-bound car has travelled, y the distance the west-bound car has travelled, and z the distance between the two cars (all after t hours). We know $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and we want to

find $\frac{dz}{dt}$. By Pythagorean theorem, we know that $z^2 = x^2 + y^2$. So we take the derivative of this with respect to time:

$$2z\left(\frac{dz}{dt}\right) = 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right)$$

Then solve for our required quantity:

$$\frac{dz}{dt} = \frac{x\left(\frac{dx}{dt}\right) + y\left(\frac{dy}{dt}\right)}{z}$$

After 2 hours, $x = (25)(2) = 50$, and $y = (60)(2) = 120$. We then can solve for z using $z^2 = x^2 + y^2$ to get $z = \sqrt{16,900}$. We then substitute all our values:

$$\frac{dz}{dt} = \frac{(50)(25) + (120)(60)}{\sqrt{16,900}} = \frac{8450}{\sqrt{16,900}}$$

Thus the two cars are going away from each other at a rate of $\frac{8450}{\sqrt{16,900}}$ km/hr.